Application of Hamiltonian Mechanics to Control Design for Industrial Robotic Manipulators

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***Abstract*—The paper deals with a tracking control for robotic manipulators, where the robot dynamics is described by means of Hamiltonian mechanics. This way leads to different physical descriptive quantities used in control design. In the paper, the model-oriented Lyapunov-based control is considered. It is introduced in the novel formulation using Hamiltonian mechanics and compared with the conventional formulation based on Lagrangian mechanics. The theoretical results, generally applicable to usual articulated industrial robotic manipulators, are demonstrated on one specific robot arm with three degrees of freedom.**

I. Introduction

Engineering practice usually employs classical vector oriented Newtonian mechanics to describe interactions of force effects. The interactions can be described by scalar functions of Lagrangian or Hamiltonian mechanics as well [1], [2]. Controllers for robots have usually to manage complicated robot structures that represent strong nonlinear systems [3], [4].

In most cases, Lagrange equations are used for the description of robot dynamics used in control design [5]. Usual state space is represented by positions and velocities, thus by kinematic quantities. Generally, there exist specific limits for positions, velocities, accelerations and control torques, respectively. They depend on a given robotic manipulator. The limits of velocities are constant for all configurations of the robot. It does not respect that the appropriate robot moments of inertia differ significantly for different configurations during the robot motion. Momenta, as adequate descriptive quantities, are not used to take these differences into account. However, just inertia moments (as momenta) change with respect to robot motion very quickly, often their rate reaches 1/10. Hence, the study of control methods from Hamiltonian point of view may be useful.

Note, in regards to Hamiltonian mechanics, that the approach employing the property of passivity of the robot was investigated, see [6]. Such approach can modify the natural energy of the robot so that it can meet the desired targets, i.e. position or tracking control. Hamiltonian mechanics was used for control design e.g. in [7], [8] and [9]. However, Hamiltonian formalism is not frequently employed. In this paper, let us investigate quite novel way, different from the aforementioned.

The aim of this paper is to answer the following question: "*Which of Lagrangian or Hamiltonian formalism is more convenient for the problems of robot control?*"

In this paper, the similar algorithms defined in Lagrangian and Hamiltonian configuration spaces will be explored and compared. The algorithms will be applied to the same problems of the robot control. Initial ideas can be found in [10].

The paper is organized as follows. Section II briefly summarizes Lagrangian formalism. Section III explains Hamiltonian formalism in details. Section IV deals with a tracking control design. Section V focuses on the mathematical comparison of usual Lagrangian versus specific Hamiltonian formalism. Finally, Section VI demonstrate by solved comparative example theoretical outputs of proposed tracking control algorithms applied to a robotic arm with three degrees of freedom.

II. Usual Lagrangian Formalism

Lagrange equations of classical Lagrangian mechanics are predominantly used for description of complicated mechanical systems [2]. These equations can be described as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = F_j, \quad j = 1, 2, \cdots, n \tag{1}$$

where n is a number of degrees of freedom (DOF); a scalar function $L = E_k - E_p$ is Lagrange function, E_k is kinetic energy and E_p is potential energy; F_j are generalized forces and q_j generalized coordinates. For technical applications, the forces F_j represent non-conservative forces. Conservative forces are represented by the potential energy E_p.

Let us consider a robotic manipulator with n DOF. The kinetic energy may be described as a quadratic positive definite form

$$E_k = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q})\,\dot{\mathbf{q}} \tag{2}$$

The potential energy can be written as

$$E_p = -\sum_{j=1}^{n} m_j \mathbf{G}^T \mathbf{T}_{\mathbf{0}}^{j} \mathbf{R}_{c,j} \tag{3}$$

If we use the equation (1), then equations of robot motion can be derived in the compact form (for details see [4], [5] and [11]):

$$\mathbf{M}(\mathbf{q})\,\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\,\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{u} \tag{4}$$

where $\mathbf{M}$ is $n \times n$ inertia matrix; $\mathbf{q}$ is $n \times 1$ vector; $\mathbf{C}$ is $n \times n$ matrix representing Coriolis and centrifugal forces; $\mathbf{g}$ is $n \times 1$ vector of gravity influences and $\mathbf{u}$ is $n \times 1$ vector of control actions relating to generalized forces $\mathbf{F}$.

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III. HAMILTONIAN FORMALISM

A. Hamilton Equations

Analytical mechanics was developed to be usable in all branches of physics. Hamilton equations have a special meaning in quantum mechanics. Forces, velocities and accelerations are not as significant for study of elementary particles as energies and momenta. Hence, let us study the meaning of the Hamiltonian formalism for the purpose to control of robotic manipulators.

For this paper aim, let a vector-matrix description be used. For instance, let generalized momentum p_j be defined as

$$p_j = \frac{\partial L}{\partial \dot{q}_j}, \quad j = 1, 2, \cdots, n \tag{5}$$

Let all vectors be defined in usual way as $\mathbf{p} = [p_1, \cdots, p_n]^T$, $\mathbf{q} = [q_1, \cdots, q_n]^T$, etc. Then, the relation (5) can be rewritten as

$$\mathbf{p} = \left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)^T \tag{6}$$

Form (6) is more suitable for further explanation. Similarly, the definition of the Hamilton function is

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L \quad \Rightarrow \quad H = \mathbf{p}^T \dot{\mathbf{q}} - L \tag{7}$$

The Lagrange function L is the function of vectors $\mathbf{q}$ and its time derivative $\dot{\mathbf{q}}$, but the Hamilton function is the function of vectors $\mathbf{q}$ and $\mathbf{p}$. So, the both functions can be generally written

$$L = L(\mathbf{q}, \dot{\mathbf{q}}, t), \quad H = H(\mathbf{q}, \mathbf{p}, t) \tag{8}$$

Then, the equations (1) can be rewritten as follows

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)^T - \left(\frac{\partial L}{\partial \mathbf{q}}\right)^T = \mathbf{F} \tag{9}$$

Similarly the equations of motion with using Hamiltonian H can be rewritten as well

$$\dot{\mathbf{q}} = \left(\frac{\partial H}{\partial \mathbf{p}}\right)^T, \quad \dot{\mathbf{p}} = \mathbf{F} - \left(\frac{\partial H}{\partial \mathbf{q}}\right)^T \tag{10}$$

The system of equations (10) are Hamilton equations in the vector form. It will be useful in further explanation.

B. Equations of Robot Dynamics

Any arbitrary robot may be considered as the time invariant system. Then, it is well known that the Hamiltonian (7) represents full energy that is the sum of kinetic and potential energies. Since the Lagrangian L depends on positions and velocities and Hamiltonian depends on positions and generalized momenta, the relations (8) can be expressed as

$$L(\mathbf{q}, \dot{\mathbf{q}}) = E_k(\mathbf{q}, \dot{\mathbf{q}}) - E_p(\mathbf{q}) \tag{11}$$

$$H(\mathbf{q}, \mathbf{p}) = E_k(\mathbf{q}, \mathbf{p}) + E_p(\mathbf{q}) \tag{12}$$

The following equation expresses the interesting fact

$$\frac{\partial E_k(\mathbf{q}, \mathbf{p})}{\partial \mathbf{q}} = -\frac{\partial E_k(\mathbf{q}, \dot{\mathbf{q}})}{\partial \mathbf{q}} \tag{13}$$

Moreover, the partial derivative of the potential energy is

$$\frac{\partial E_p}{\partial \mathbf{q}} = \mathbf{g}^T(\mathbf{q}) \tag{14}$$

Equations, describing the dynamics or robot motion, can be defined as follows [10]

$$\dot{\mathbf{q}} = \mathbf{M}^{-1}\mathbf{p} \tag{15}$$

$$\dot{\mathbf{p}} = \mathbf{F} - \mathbf{g}(\mathbf{q}, t) - \left(\frac{\partial E_k(\mathbf{q}, \mathbf{p})}{\partial \mathbf{q}}\right)^T \tag{16}$$

The kinetic energy (2) in $(\mathbf{q}, \mathbf{p})$ coordinates is as follows

$$E_k(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\mathbf{p}^T \mathbf{M}^{-1}(\mathbf{q})\,\mathbf{p} \tag{17}$$

Thus, the relations (15) and (16) fully describe the robot dynamics in the Hamiltonian formalism.

C. Simplification for Control Design

Let us consider a specific skew symmetric matrix $\mathbf{S}$ defined by components as

$$S_{ij} = \frac{1}{2}\sum_{k=1}^{n} \dot{q}_k \left(\frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{jk}}{\partial q_i}\right) \tag{18}$$

Then, it can be proved that

$$\mathbf{S}\,\dot{\mathbf{q}} = \frac{1}{2}\dot{\mathbf{M}}\,\dot{\mathbf{q}} - \frac{1}{2}\left(\frac{\partial}{\partial \mathbf{q}}\left(\dot{\mathbf{q}}^T\,\mathbf{M}\,\dot{\mathbf{q}}\right)\right) \tag{19}$$

The equations (15) and (16) can be expressed in the following compact form [10]

$$\dot{\mathbf{q}} = \mathbf{M}^{-1}\mathbf{p} \tag{20}$$

$$\dot{\mathbf{p}} = \left(\frac{1}{2}\dot{\mathbf{M}} - \mathbf{S}\right)\mathbf{M}^{-1}\mathbf{p} - \mathbf{g} + \mathbf{u} \tag{21}$$

This form is more suitable for the study of control stability. The vector $\mathbf{F}$ in (16) is replaced in (21) by $\mathbf{u}$. Note that all velocities in (21) must be rewritten by (20), since now $(\mathbf{q}, \mathbf{p})$ coordinates are considered. The vector $\mathbf{u}$ in (21) plays a role of a control vector similarly as in (4). The equations (20) and (21) are the final equations that describe the dynamics of robot motion.

The conditions are derived in [10]. Their meeting leads to better robot positioning when the Hamiltonian formalism is used for control design. In the following section, the tracking control, i.e. control along a desired trajectory will be studied.

IV. TRACKING CONTROL

The tracking control is a standard task of robot control along required trajectories. For articulated robots, the trajectory is usually given by the time sequence of joint coordinates and their appropriate derivatives. Several schemes for performing these objectives exist, e.g. well-known inverse dynamic control and computed torque control [4] or the passivity based control and the Lyapunov-based control [5].

Here, one of Lyapunov-based control with state transformation will be introduced. It will be shown just as exponentially stable control according to Lyapunov theory of stability. Similar algorithm was derived in [12]. The derivation was based on classical Lagrangian formalism.

For simplicity, we shall call the *Hamiltonian space* the space defined by coordinates $(\mathbf{q}, \mathbf{p})$. Similarly, the *Lagrange space* will be the space with coordinates given by positions $\mathbf{q}$ and their time derivatives $\dot{\mathbf{q}}$. Note that these definitions are not commonly used, but, for the paper aim, they will be useful.

A. Tracking Control in Hamilton Space

Let the following vector transformations be considered

$$\mathbf{e} = \mathbf{q} - \mathbf{q}_d , \quad \mathbf{z} = \mathbf{M}\,(\dot{\mathbf{e}} - \mathbf{A}\mathbf{e}) , \quad \mathbf{y} = \mathbf{p} - \mathbf{z} \tag{22}$$

where $\mathbf{e}$ is a control error and $\dot{\mathbf{e}}$ is its appropriate derivative. Then, the robotic system (20) and (21) can be controlled by the following control law

$$\mathbf{u} = \dot{\mathbf{y}} - \left(\frac{1}{2}\dot{\mathbf{M}} - \mathbf{S}\right)\mathbf{M}^{-1}\mathbf{y} + \mathbf{g} - \mathbf{B}\,\mathbf{z} \tag{23}$$

where $\mathbf{A}$ and $\mathbf{B}$ are non-singular matrices of control parameters; $\mathbf{y}$ represents the estimation of the momentum $\mathbf{p}$; and $\mathbf{z}$ is a difference between current momentum $\mathbf{p}$ and its estimate $\mathbf{y}$. By the insertion of (23) into (21), the feedback equation is:

$$\dot{\mathbf{z}} = \left(\frac{1}{2}\dot{\mathbf{M}} - \mathbf{S}\right)\mathbf{M}^{-1}\mathbf{z} - \mathbf{B}\,\mathbf{z} \tag{24}$$

It represents expected trend of $\mathbf{z}$ used for determination of $\mathbf{y}$.

B. Exploration of the Stability Conditions

To explore stabilizing control actions, let us define a positive definite quadratic form

$$W = \frac{1}{2}\mathbf{z}^T\mathbf{M}^{-1}\,\mathbf{z} \tag{25}$$

where its time derivative, considering (24), leads to

$$\dot{W} = -\mathbf{z}^T\mathbf{M}^{-1}\mathbf{B}\,\mathbf{z} \;\leq\; 0 \tag{26}$$

The multiplication of matrices in the quadratic form (26) is positive definite. So the function W decreases in time. Then, the following inequality from (26) can be obtained

$$0 \;\leq\; \int_0^t \mathbf{z}^T\mathbf{M}^{-1}\mathbf{B}\,\mathbf{z}\,dt \;=\; -\int_0^t \dot{W}\,dt \;=\; W(0) - W(t) \;\leq\; W(0) \tag{27}$$

Let $\lambda_{\min}$ be the smallest eigenvalue of the matrix $\mathbf{M}^{-1}\mathbf{B}$, i.e.

$$\lambda_{\min} = \min\left\{\lambda_q ;\; \mathbf{M}^{-1}(\mathbf{q})\mathbf{B}\,\mathbf{x}(\mathbf{q}) = \lambda_q\mathbf{x}(\mathbf{q}), \mathbf{q} \in Q\right\} \tag{28}$$

where Q is a domain of the robot. For $\lambda_{\min} > 0$ and all $\mathbf{z}$, the following inequality is valid

$$\lambda_{\min}\mathbf{z}^T\mathbf{z} \;\leq\; \mathbf{z}^T\mathbf{M}^{-1}\mathbf{B}\,\mathbf{z} \tag{29}$$

This relation implies the following inequality

$$\int_0^t \mathbf{z}^T\mathbf{z}\,dt \;=\; \frac{1}{\lambda_{\min}}\int_0^t \mathbf{z}^T\mathbf{M}^{-1}\mathbf{B}\,\mathbf{z}\,dt \;\leq\; \frac{W(0)}{\lambda_{\min}} \;<\; \infty \tag{30}$$

Since $\|\mathbf{z}\|^2 = \mathbf{z}^T\mathbf{z}$ and $0 \leq W(t) \leq W(0)$, then it can be seen that $\mathbf{z} \in L_2 \cap L_\infty$, [13]. Similarly to (28), let us define the following eigenvalues

$$\lambda_m = \min\left\{\lambda_q ;\; \mathbf{M}(\mathbf{q})\,\mathbf{x}(\mathbf{q}) = \lambda_q\mathbf{x}(\mathbf{q}),\; \mathbf{q} \in Q\right\} \tag{31}$$

$$\lambda_M = \max\left\{\lambda_q ;\; \mathbf{M}(\mathbf{q})\,\mathbf{x}(\mathbf{q}) = \lambda_q\mathbf{x}(\mathbf{q}),\; \mathbf{q} \in Q\right\} \tag{32}$$

The values (31) and (32) are positive as well. They can be involved in the following inequalities

$$\lambda_m\mathbf{z}^T\mathbf{z} \;\leq\; \mathbf{z}^T\mathbf{M}\,\mathbf{z} \;\leq\; \lambda_M\mathbf{z}^T\mathbf{z} \tag{33}$$

These inequalities with quadratic forms are valid for all $\mathbf{z}$. Hence, they can be simply written as

$$\lambda_m\mathbf{I} \;\leq\; \mathbf{M}(\mathbf{q}) \;\leq\; \lambda_M\mathbf{I} \tag{34}$$

It can be proved that inequalities (34) can be expressed as

$$\lambda_M^{-1}\,\mathbf{I} \;\leq\; \mathbf{M}^{-1}(\mathbf{q}) \;\leq\; \lambda_m^{-1}\,\mathbf{I} \tag{35}$$

Then, new inequality can be introduced

$$\frac{\dot{W}}{W} \;=\; -2\frac{\mathbf{z}^T\mathbf{M}^{-1}\mathbf{B}(\mathbf{q})\,\mathbf{z}}{\mathbf{z}^T\mathbf{M}^{-1}(\mathbf{q})\,\mathbf{z}} \;\leq\; -2\frac{\mathbf{z}^T\mathbf{z}\,\lambda_{\min}}{\mathbf{z}^T\mathbf{z}\,\lambda_m^{-1}} \tag{36}$$

It leads to the following compact inequality

$$\frac{\dot{W}}{W} \;\leq\; -2\lambda_{\min}\lambda_m \tag{37}$$

Let the multiplication of the eigenvalues be denoted as follows $a = \lambda_{\min}\lambda_m$. Then, inequality (37) may be rewritten as

$$\frac{\dot{W}}{W} \;\leq\; -2a \tag{38}$$

Integration of the inequality (38) leads to the estimation

$$W(t) \leq W(0)\,e^{-2at} \tag{39}$$

Then, the following expressions from (25) and (35) can be done

$$\begin{aligned} \lambda_M^{-1}\mathbf{z}^T\mathbf{z} \;&\leq\; \mathbf{z}^T\mathbf{M}^{-1}\,\mathbf{z} \;=\; 2W \;\leq\; 2W(0)\,e^{-2at} \\ &= \mathbf{z}^T(0)\mathbf{M}^{-1}(q_0)\,\mathbf{z}(0)\,e^{-2at} \;\leq\; \|\,\mathbf{z}(0)\,\|^2\;\lambda_m^{-1}e^{-2at} \end{aligned} \tag{40}$$

Hence, the variable $\mathbf{z}$ is exponentially bounded from above

$$\|\mathbf{z}(t)\| \le c_1 e^{-at} \quad \text{for} \quad c_1 = (\lambda_M \lambda_m^{-1})^{0.5} \|\mathbf{z}(0)\| \qquad (41)$$

From (41), the variable $\mathbf{z} \to \mathbf{0}$ for $t \to \infty$. Let us study the differential equation obtained from (22)

$$\dot{\mathbf{e}} - \mathbf{A}\,\mathbf{e} = \mathbf{M}^{-1}\mathbf{z} \qquad (42)$$

This equation has a solution starting from $\mathbf{e}_0 = \mathbf{e}(0)$

$$\mathbf{e}(t) = \exp(\mathbf{A}t)\,\mathbf{e}_0 + \int_0^t \exp(\mathbf{A}(t-\tau))\,\mathbf{M}^{-1}\mathbf{z}\,d\tau \qquad (43)$$

Using inequalities for matrices and vectors with norms, the following estimation can be derived (derivation can be found in Appendix A):

$$\|\mathbf{e}(t)\| \le c_2 e^{-bt} \qquad (44)$$

for some positive constants c_2 and b and for $t \to \infty$ the error vector $\mathbf{e} \to \mathbf{0}$ as well.

Let us rewrite the equation (42) in the form

$$\dot{\mathbf{e}} = \mathbf{M}^{-1}\,\mathbf{z} + \mathbf{A}\,\mathbf{e} \qquad (45)$$

then

$$\|\dot{\mathbf{e}}\| \le \|\mathbf{M}^{-1}\| \cdot \|\mathbf{z}\| + \|\mathbf{A}\| \cdot \|\mathbf{e}\| \qquad (46)$$

Using (41) and (44), the following inequality is obtained:

$$\|\dot{\mathbf{e}}(t)\| \le c_3 e^{-bt} \qquad (47)$$

where c_3 is a positive constant. Hence, if $t \to \infty$, then the velocity of the error signal $\dot{\mathbf{e}}$ converges to $\mathbf{0}$.

The relations (46) and (47) show that the control algorithm defined in the section IV-A is exponentially stable.

C. *Selection of Matrices* A *and* B

To ensure stability conditions, the matrix $\mathbf{A}$ has to be stable, i.e. its eigenvalues have to be in the left side of the complex plane. The matrix product $\mathbf{M}^{-1}\,\mathbf{B}$ must be positive definite. Since $\mathbf{M}$ and $\mathbf{M}^{-1}$ are positive definite matrices, it is sufficient so that $\mathbf{B}$ be diagonal matrix with positive elements. Then, the product $\mathbf{M}^{-1}\,\mathbf{B}$ is positive definite as well.

Note that the multiplication of arbitrary two positive definite matrices does not be a positive definite matrix again.

For simplicity, the matrix $\mathbf{B}$ can be selected as $\mathbf{B} = b_0\,\mathbf{I}$. Any alternative choice is to define some matrix $\mathbf{B}_0$ as positive definite and then define matrix $\mathbf{B}$ as $\mathbf{B} = \mathbf{M}(\mathbf{q})\,\mathbf{B}_0$. Then, $\mathbf{M}^{-1}\,\mathbf{B} = \mathbf{B}_0$ in (26) is automatically the positive definite matrix.

D. *Tracking Control in Lagrange Space*

Similarly to the previous part, let us define new vectors $\mathbf{y}$, $\mathbf{v}$

$$\mathbf{e} = \mathbf{q} - \mathbf{q}_d\,, \quad \mathbf{v} = \dot{\mathbf{e}} - \mathbf{A}\mathbf{e}\,, \quad \dot{\mathbf{y}} = \dot{\mathbf{q}} - \mathbf{v} \qquad (48)$$

and consider the control law as follows

$$\mathbf{u} = \mathbf{M}\,\ddot{\mathbf{y}} + \mathbf{C}\,\dot{\mathbf{y}} + \mathbf{g} - \mathbf{B}\,\mathbf{v} \qquad (49)$$

If this control law is substituted into (4), then the feedback is:

$$\mathbf{M}\,\dot{\mathbf{v}} + \mathbf{C}\,\mathbf{v} + \mathbf{B}\,\mathbf{v} = \mathbf{0} \qquad (50)$$

Similarly to (25), let us define quadratic form

$$V = \frac{1}{2}\mathbf{v}^T\mathbf{M}\,\mathbf{v} \qquad (51)$$

where its time derivation is

$$\dot{V} = -\mathbf{v}^T\mathbf{B}\,\mathbf{v} \le 0 \qquad (52)$$

Hence, we have proved the following exponential stability theorem of robot control.

Theorem 1.
The discussed control laws given by (22) *and* (23) *have the following properties:*

1. $\mathbf{z} \in L_2 \cap L_\infty$, $\mathbf{e} \in L_2 \cap L_\infty$, $\dot{\mathbf{e}} \in L_2 \cap L_\infty$

2. $\|\mathbf{z}(t)\| \le c_1 e^{-at}$, $\|\mathbf{e}(t)\| \le c_2 e^{-bt}$, $\|\dot{\mathbf{e}}(t)\| \le c_3 e^{-bt}$

where a, b, c_1, c_2, c_3 are positive constants.

V. COMPARISON: HAMILTON VERSUS LAGRANGE

This section compares the convergence velocity of the control from two points of view. Let us compare (25) and (51). The relation (25) can be expressed as

$$W = \frac{1}{2}(\mathbf{M}\mathbf{v})^T\mathbf{M}^{-1}\,\mathbf{M}\mathbf{v} = \frac{1}{2}\mathbf{v}^T\mathbf{M}\mathbf{v} = V \qquad (53)$$

Now, let us compare (26) and (52). The relation (26) can be rewritten as

$$\dot{W} = -(\mathbf{M}\mathbf{v})^T\,\mathbf{M}^{-1}\mathbf{B}\,\mathbf{M}\,\mathbf{v} = -\mathbf{v}^T\mathbf{B}\,\mathbf{M}\,\mathbf{v} \qquad (54)$$

Let us study the conditions for which the following inequality is valid

$$\dot{W} < \dot{V} \qquad (55)$$

From (52) and (54), it can be seen that inequality (55) is valid if and only if

$$0 < \mathbf{v}^T\mathbf{B}(\mathbf{M} - \mathbf{I})\,\mathbf{v} \qquad (56)$$

This is true for $\mathbf{v} \ne \mathbf{0}$ if $\mathbf{B}\,(\mathbf{M}–\mathbf{I})$ is a positive definite matrix. Since $\mathbf{B}$ is a diagonal positive definite matrix, it sufficient so that $\mathbf{M}–\mathbf{I}$ be positive definite matrix.

Theorem 2.
If $\mathbf{M} - \mathbf{I}$ *is a positive definite matrix and the control of robot uses the access introduced in the section IV.A, then this control process is faster than the similar control process described in the section IV-D, if it starts from the same initial error* $\mathbf{e}$ *and appropriate time derivative* $\dot{\mathbf{e}}$.

Note that this condition is true for considered industrial robots.

VI. Illustrative Example

Let us consider the robotic arm with 3 DOF, the kinematic structure of which is shown in the Fig. 1. It corresponds to configuration of the industrial robot JANOME.

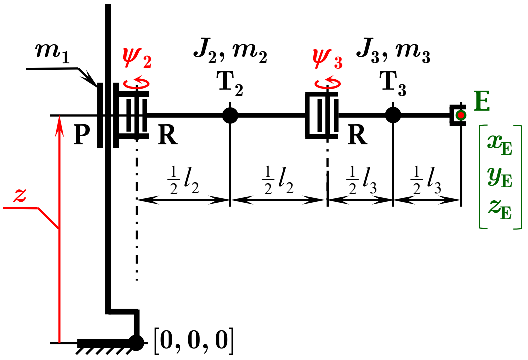

Fig. 1. Structure of robotic arm with 3 DOF.

The robot arm represents kinematic configuration containing one prismatic (P) and two revolute (R) joints. Its model, considering masses m_i and moments of inertia J_i in centres of gravity T_i, is composed with the following elements:

$$\mathbf{q} = \begin{bmatrix} z \\ \varphi_2 \\ \varphi_3 \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} mg \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & M_{22} & M_{23} \\ 0 & M_{23} & M_{33} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & S_{23} \\ 0 & -S_{23} & 0 \end{bmatrix}$$

for total mass $m = m_1 + m_2 + m_3$, element $S_{23} = 2K_a(\dot{\psi}_2 + \frac{1}{2}\dot{\psi}_3)$ of the skew symmetric matrix $\mathbf{S}$ and the following variables

$$K_a = \frac{1}{4} l_2 l_3 m_3 \cos\varphi_3, \quad K_b = \frac{1}{4} l_2 l_3 m_3 \sin\varphi_3, \quad \varphi_3 = \psi_3 + \frac{\pi}{2}$$

$$M_{33} = J_3 + \frac{1}{4} l_3^2 m_3, \quad M_{23} = M_{33} + K_b$$

$$M_{22} = J_2 + l_2^2 (\frac{1}{4} m_2 + m_3) + M_{33} + 2K_b$$

From (20), velocities is obtained as follows

$$\begin{aligned} \dot{z}_1 &= p_1/m \\ \dot{\psi}_2 &= (M_{33}p_2 - M_{23}p_3)/F \quad , \qquad F = M_{22}M_{33} - M_{23}^2 \\ \dot{\psi}_3 &= (M_{22}p_3 - M_{23}p_2)/F \end{aligned}$$

where the momenta has to be substituted via the velocities. The equation (21) can be described in a scalar form as

$$\begin{aligned} &\dot{p}_1 = u_1 - mg \\ &\dot{p}_2 = u_2 \\ &\dot{p}_3 = u_3 - K_a F^{-1}(1 + 2K_b F^{-1} M_{23}) p_2 p_3 - K_a F^{-1}(1 + K_b F^{-1} M_{22}) p_3^2 \end{aligned}$$

The condition, that the matrix $\mathbf{M} - \mathbf{I}$ is positive definite, is equivalent to the inequality conditions $m > 1$, $M_{22} > 1$ and $F > M_{22} + M_{33} - 1$.

The following figures Fig. 2 - Fig. 4 show the robot motion along reference trajectory and control errors for ψ_2 and ψ_3.

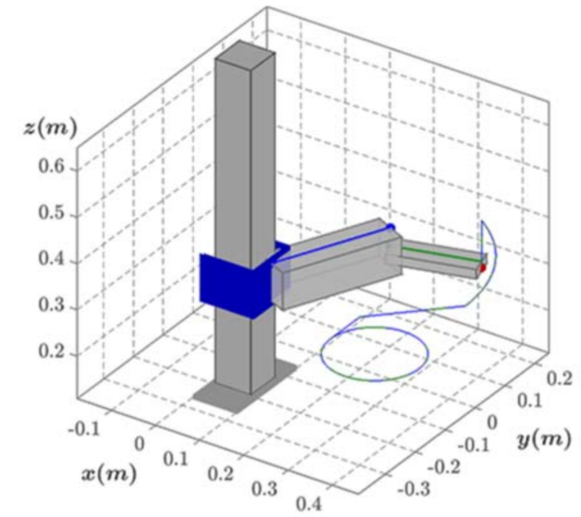

Fig. 2. 3D model of robotic arm and testing trajectory.

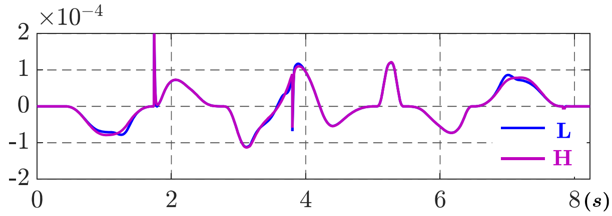

Fig. 3. Comparison of errors of the coordinate q_2 (*rad*).

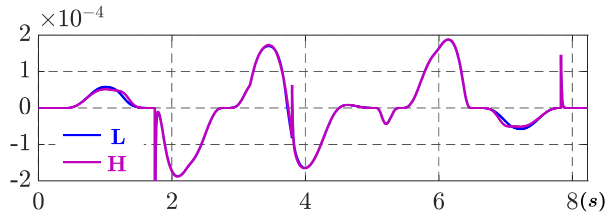

Fig. 4. Comparison of errors of the coordinate q_3 (*rad*).

Note, that error in q_1 is not pictured due to the independence of q_1 of other coordinates, as well as corresponding control actions.

Fig. 5 shows corresponding kinematic quantitates of the testing trajectory. Fig. 6 shows comparison of the appropriate control actions. It is obvious the smaller magnitudes especially of the third control action related to the coordinate ψ_3. Fig. 7 shows the similar result showing better energy distribution in Hamiltonian space, but compared with cumulative control actions, where it is obvious especially for control action u_3. Hence, for comparable control errors and smaller control actions, the Hamiltonian formalism shows more suitable behavior in comparison with usual Lagrangian description. It is significant and promising for robot motion optimization including an adequate internal energy distribution.

VII. Conclusion

In this paper, the novel method of tracking control expressed in *Hamilton coordinates* ($\mathbf{q}$, $\mathbf{p}$) is introduced. In section V, it was proved that this control is faster than classical control in state space ($\mathbf{q}$, $\dot{\mathbf{q}}$). This result is true generally for almost all control methods applied to electro dynamical systems, where the momenta quickly change their values.

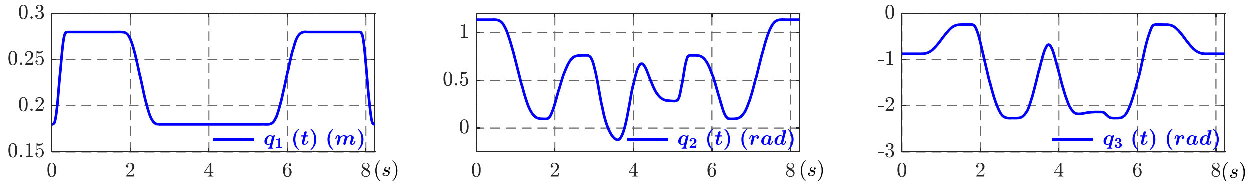

Fig. 5. Time histories of kinematic quantitates: $q_1 = z$; $q_2 = \psi_2$; $q_3 = \psi_3$.

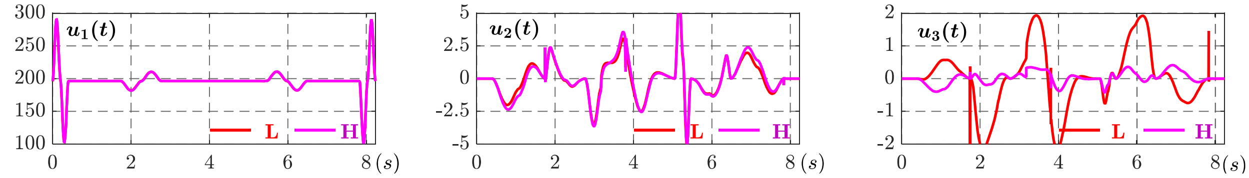

Fig. 6. Time histories of control actions for Lagrangian and Hamiltonian formalisms: u_1 (N); u_2 ($N{\cdot}m$); u_3 ($N{\cdot}m$).

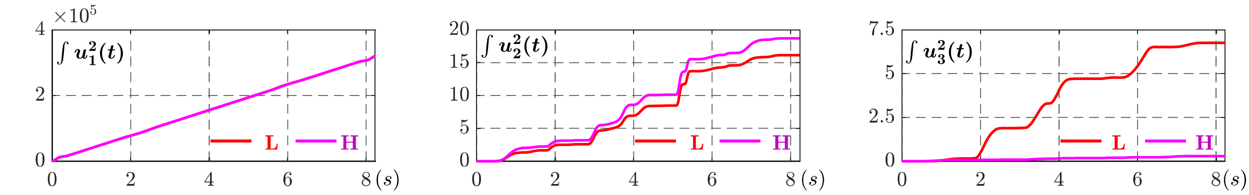

Fig. 7. Time histories of cumulative control actions for Lagrangian formalism (left) and Hamiltonian formalism (right).

The advantage is that control algorithms can be simply expressed in the new space $\mathbf{q}$ and $\mathbf{p}$. A drawback is that the vector of momenta $\mathbf{p}$ has to be computed, since only velocities can be measured. But the state space ($\mathbf{q}$, $\dot{\mathbf{q}}$) represents only kinematic variables that do not represent robot dynamics at all. The space ($\mathbf{q}$, $\mathbf{p}$) represents positions and dynamical parameters $\mathbf{p}$. Hence, from the control point of view, the Hamilton equations are more convenient for electromechanical systems with more DOF.

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Appendix A. Proof of the Equation (44)

Let the solution (43) be considered. Then, it can be derived

$$\|\mathbf{e}(t)\| \le \|\exp(\mathbf{A}t)\|\cdot\|\mathbf{e}_0\| + \int_0^t \|\exp(\mathbf{A}(t-\tau))\|\cdot\|\mathbf{M}^{-1}\|\cdot\|\mathbf{z}\|\,d\tau \quad (57)$$

Since $\mathbf{A}$ is stable matrix, there exist positive constants k and c for $s \ge 0$ in the following expression

$$\|\exp(\mathbf{A}s)\| \le ke^{-cs} \quad (58)$$

where e is the Euler number. Thus, (57) may be expressed as

$$\|\mathbf{e}(t)\| \le k\|\mathbf{e}_0\|e^{-ct} + k\int_0^t e^{c(\tau-t)}\|\mathbf{M}^{-1}\|\cdot\|\mathbf{z}\|\,d\tau \quad (59)$$

The domain Q of all admissible vectors $\mathbf{q}$ is bounded, there exists constant k_M such that

$$\|\mathbf{M}(\mathbf{q})\| \le k_M \quad (60)$$

Hence with using these facts and (41), the following expression can be written

$$\|\mathbf{e}(t)\| \le k\|\mathbf{e}_0\|e^{-ct} + c_1\,k\,k_M e^{-ct}\int_0^t e^{\tau(c-a)}\,d\tau \quad (61)$$

If $c = a$, then the integral in (61) is t. Generally it is $c \ne a$. However, the case $c = a$ can be included in this second case, if c is replaced by any smaller positive c. Then

$$\|\mathbf{e}(t)\| \le k\|\mathbf{e}_0\|e^{-ct} + c_1 k\,k_M\frac{e^{-at}-e^{-ct}}{c-a} \quad (62)$$

Let $b = \min\{a, c\}$. Then $0 \le \dfrac{e^{-at}-e^{-ct}}{c-a} = \dfrac{e^{-bt}}{|c-a|}$ and (62) can be expressed as $\|\mathbf{e}(t)\| \le e^{-bt}\left[k\|\mathbf{e}_0\| + \dfrac{c_1\,k\,k_M}{|c-a|}\right]$ (63)

Hence (63) has the form of (44).